

On piecewise pluriharmonic functions*

B. Kazarnovskii

We extend some results on piecewise linear functions on \mathbb{C}^n to piecewise pluriharmonic functions on any complex manifold. We construct a ring generated by currents h and $dd^c h$, where $\{h\}$ is a finite set of piecewise pluriharmonic functions. We prove that, with some restrictions on the set $\{h\}$, the map $\{h \mapsto dd^c h, dd^c h \mapsto 0\}$ can be continued to the derivation on the ring. As a corollary, the current $dd^c g_1 \wedge \cdots \wedge dd^c g_k$ depends on the product of piecewise pluriharmonic functions g_1, \dots, g_k only.

1 Results.

A function $g: M \rightarrow \mathbb{R}$ on a complex manifold M is called pluriharmonic if $dd^c g = 0$ (recall that $d^c g(x_t) = dg(\sqrt{-1}x_t)$). Another definition: the function g is a real part of some holomorphic function in some neighborhood of any point $x \in M$.

Definition 1. Continuous function $g: M \rightarrow \mathbb{R}$ on n -dimensional complex manifold M is called *piecewise pluriharmonic* (or PPH-function) if g is pluriharmonic on any closed $2n$ -dimensional simplex of some locally finite triangulation of the manifold M (see subsection 2.1).

Piecewise linear functions on the space \mathbb{C}^n are the simplest examples of PPH-functions. Piecewise linear functions are used in convex geometry, algebraic geometry, and complex analysis [2]-[9]. We extend some results on piecewise linear functions on \mathbb{C}^n [2] to PPH-functions on any complex manifold.

There exists a nonconstant PPH-function on any complex manifold. Indeed, let h be a piecewise linear function on the space \mathbb{C}^n and $h = 0$ outside some small neighborhood of zero. Let H be a function on M such that $H = h$ on the coordinate neighborhood of some point $x \in M$ and $H = 0$ outside this neighborhood. The function H is piecewise pluriharmonic.

*Supported by NSh-4850.2012.1

Definition 2. Let A be some vector space consisting of PPH-functions on an n -dimensional complex manifold M . The vector space A is called *the constructive space* if for any finite set of elements $H = \{h_i \in A\}$ there exists a triangulation \mathcal{P}_H of M such that each function h_i is pluriharmonic on every closed $2n$ -dimensional simplex $\Delta \in \mathcal{P}_H$.

In the sequel, all PPH-functions are the elements of some fixed constructive space A .

Example 1. The space of piecewise linear functions on \mathbb{C}^n is the constructive space.

Question 1. Is the space of all PPH-functions constructive?

Let $h_i: M \rightarrow \mathbb{R}$ be a function on a complex n -dimensional manifold M . The mixed Monge-Ampere operator on M of degree k is (by definition) the map $(h_1, \dots, h_k) \mapsto dd^c h_1 \wedge \dots \wedge dd^c h_k$. If h_1, \dots, h_k are continuous plurisubharmonic functions, then [10] the Monge-Ampere operator value $dd^c h_1 \wedge \dots \wedge dd^c h_k$ is well defined as a current (that is a functional on the space of smooth compactly supported differential $(2n - 2k)$ -forms). This means that if the sequence of smooth plurisubharmonic functions $(f_i)_j$ converges locally uniformly to h_i , then the sequence of currents $dd^c(f_1)_j \wedge \dots \wedge dd^c(f_k)_j$ converges to the limit current, independent of the choice of approximation. This limit current is the current of measure type, ie may be continued to a functional on the space of continuous compactly supported forms. It follows that any polynomial in the variables $h_1, \dots, h_p, dd^c g_1, \dots, dd^c g_q$ with continuous plurisubharmonic functions h_i and g_i gives the well defined current on M .

It is easy to prove that any PPH-function can be locally written as a difference of two continuous plurisubharmonic functions. It follows that above defined currents are well defined for PPH-functions $h_1, \dots, h_p, g_1, \dots, g_q$ also.

Theorem 1. Let \mathcal{A} be a ring of currents, generated by currents h and $dd^c h$ with $h \in A$. There exists a derivation δ on the ring \mathcal{A} such that $\delta(h) = dd^c h$ and $\delta(dd^c h) = 0$ for any PPH-function $h \in A$.

Remark 1. Let h_i be PPH-functions and $F = h_0 dd^c h_1 \wedge \dots \wedge dd^c h_k$; then $\delta(F) = dd^c F$. But $\delta(h^2) = 0$ and $dd^c(h^2) \neq 0$ for any nonconstant pluriharmonic function h .

Corollary 1. $\delta^k(h_1 \dots h_k) = k! dd^c h_1 \wedge \dots \wedge dd^c h_k$ for any PPH-functions h_1, \dots, h_k .

Corollary 2. *The current $dd^c h_1 \wedge \cdots \wedge dd^c h_k$ depends on the product of PPH-functions h_1, \dots, h_k only.*

Corollary 3. *Let the space A , generated by the PPH-functions h_1, \dots, h_k , be a constructive space and let $h = \max(h_1, \dots, h_k)$. Suppose that $h \in A$. Then $dd^c(h - h_1) \wedge \cdots \wedge dd^c(h - h_k) = 0$.*

The proof of theorem 1 uses (as the proofs of similar theorems for piecewise linear functions in [1, 2]) a construction of k -th corner locus. Such corner loci (in some more special situation) were constructed in [1].

2 PPH-cycles and corner loci.

2.1 P-cycles and operator D_c .

A simplex Δ in a complex n -dimensional manifold M is the image of a smooth nonsingular embedding $\Delta^{2n} \rightarrow M$, where Δ^{2n} is the standard closed $2n$ -dimensional simplex. A locally finite set \mathcal{P} of simplices in M is called a *triangulation* if $\cup_{\Delta \in \mathcal{P}} \Delta = M$ and intersection of any two simplices from \mathcal{P} is their common (may be empty) face. Any face of any simplex $\Delta \in \mathcal{P}$ is called a *cell* of triangulation \mathcal{P} . An odd form ω on the cell Δ is called the *frame* of Δ (recall that the odd form on a manifold with orientations α, β is a pair of forms $\omega_\alpha, \omega_\beta$ such that $\omega_\alpha = -\omega_\beta$). Let k -dimensional chain (or k -chain) be a map $X: \Delta \mapsto X_\Delta$ of the set of all k -dimensional cells to their frames. Let the map ∂ take each k -chain X to $(k-1)$ -chain ∂X , where

$$(\partial X)_\Lambda = \sum_{\Delta \supset \Lambda, \dim \Delta = k} X_\Delta,$$

where the orientations of the cells Λ and Δ agreed as usual. A k -chain X is called a k -cycle if $\partial X = 0$.

Corollary 4. *For any k -chain X the $(k-1)$ -chain ∂X is a cycle.*

In the sequel we assume that $k \geq n$.

Let Δ, T_Δ^x , and \mathbb{C}_Δ^x be (respectively) a k -dimensional cell, the tangent space of Δ at the point $x \in \Delta$, and the maximal complex subspace of T_Δ^x . Say that the point $x \in \Delta$ is nondegenerate if $\text{codim}_{\mathbb{C}} \mathbb{C}_\Delta^x = \text{codim} T_\Delta^x$. For $k = 2n, 2n-1$ any point of k -dimensional cell is nondegenerate. If x is nondegenerate, then $\dim T_\Delta^x - \dim_{\mathbb{R}} \mathbb{C}_\Delta^x = 2n - k$. If x is degenerate, then the form X_Δ (from definition 3) is zero at x .

Definition 3. A k -cycle X is called P -cycle if the following conditions hold:

- (1) $\deg X_\Delta = 2n - k$
- (2) $X_\Delta = \sum_{1 \leq j \ll \infty} g_\Delta^j W_\Delta^j$, where $(2n - k)$ -forms W_Δ^j are closed
- (3) $W_\Delta(\xi_1, \dots, \xi_{2n-k}) = 0$ for any $x \in \Delta$ and any $(\xi_1, \dots, \xi_{2n-k}) \subset T_\Delta^x$ if $\exists i: \xi_i \in \mathbb{C}_\Delta^x$.

Let X be a k -dimensional P -cycle. Define the $(k - 1)$ -cycle $D_c X = \partial Y$, where Y is a k -chain such that

$$Y_\Delta = \sum_{1 \leq j \ll \infty} d^c G_\Delta^j \wedge W_\Delta^j, \quad (1)$$

where G_Δ^j are smooth functions in some neighborhood of Δ such that $G_\Delta^j(x) = g_\Delta^j(x)$ for any $x \in \Delta$.

Corollary 5. If $k = n$ then $D_c X = 0$.

Proposition 1. (a) The $(k - 1)$ -cycle $D_c X$ does not depend on the choice of functions G_Δ^j in (1) and on the choice of functions g_Δ^j in decomposition $X_\Delta = \sum_{1 \leq j \ll \infty} g_\Delta^j W_\Delta^j$ from definition 3.

(b) $D_c X_\Lambda(\xi_1, \dots, \xi_{2n-k+1}) = 0$ for $x \in \Lambda$ and $(\xi_1, \dots, \xi_{2n-k+1}) \subset T_\Lambda^x$ if $\exists i: \xi_i \in \mathbb{C}_\Lambda^x$.

Proof. If $F(x) = 0$ for any $x \in \Delta$, then the form $d^c F$ is zero on the subspace $\mathbb{C}_\Delta^x \subset T_\Delta^x$. Using definition 3 (3) we get that the form $d^c F \wedge W_\Delta^j$ is zero on Δ . It follows that the form $(D_c X)_\Delta$ does not depend on the choice of functions G_Δ^j .

Now we prove that the form $(D_c X)_\Lambda$ does not depend on the choice of functions g_Δ^j . Let $x_1 \wedge \dots \wedge x_{2n-k+1} \neq 0$, where $x_i \in T_\Delta^x$ and $x_1 \in \mathbb{C}_\Delta^x$. If $(\sum_j g_\Delta^j W_\Delta^j) = 0$, then

$$\begin{aligned} \sum_j d^c G_\Delta^j \wedge W_\Delta^j(x_1, x_2, \dots) &= \sum_j dG_\Delta^j \wedge W_\Delta^j(ix_1, x_2, \dots) = \\ &= \sum_j dg_\Delta^j \wedge W_\Delta^j(ix_1, x_2, \dots) = d \left(\sum_j g_\Delta^j \wedge W_\Delta^j \right) (ix_1, x_2, \dots) = 0. \end{aligned}$$

The first equality follows from the definition of operator d^c and from definition 3, the third – from the closedness of forms W_Δ^j . Assertion (a) is proved.

Let $\dim \Lambda = k - 1$ and let $x_1 \wedge \dots \wedge x_{2n-k+1} \neq 0$, where $x_i \in T_\Lambda^x$ and

$x_1 \in \mathbb{C}_\Lambda^x$. We prove that $(D_c X)_\Lambda(x_1, \dots, x_{2n-k+1}) = 0$.

$$\begin{aligned} (D_c X)_\Lambda(x_1, x_2, \dots) &= \sum_{\Delta \supset \Lambda, \dim \Delta = k} \sum_j d^c G_\Delta^j \wedge W_\Delta^j(x_1, x_2, \dots) = \\ &= \sum_{\Delta \supset \Lambda, \dim \Delta = k} \sum_j dG_\Delta^j \wedge W_\Delta^j(ix_1, x_2, \dots) = \sum_{\Delta \supset \Lambda, \dim \Delta = k} dX_\Delta(ix_1, x_2, \dots) = \\ &= d \left(\sum_{\Delta \supset \Lambda, \dim \Delta = k} X_\Delta(ix_1, x_2, \dots) \right) = d(\partial X)_\Lambda = 0. \end{aligned}$$

The first equality is a definition of $(k-1)$ -chain ∂Y . Other equalities follow from the definition of operator d^c , the closedness of forms W_Δ^j , and the closedness of the k -chain X . \square

Remark 2. $D_c X$ is not necessarily a P-cycle.

2.2 Corner loci of PPH-polynomials.

Definition 4. Let $H = \{h_1, \dots, h_q\}$ be a basis of the constructive space A , $P(x_1, \dots, x_q)$ be a polynomial of degree m in the variables x_1, \dots, x_q . The function $P(h_1, \dots, h_q)$ is called a PPH-polynomial of degree m .

The degree of PPH-polynomial is not uniquely defined. But PPH-polynomials of degree 0 are constants and PPH-polynomials of degree 1 are PPH-functions.

Below we fix the set H and the triangulation \mathcal{P}_H . The restriction of PPH-polynomial P to standardly oriented $2n$ -cells of triangulation \mathcal{P}_H give the $2n$ -dimensional P-cycle X^P . Say that $(2n-1)$ -cycle $D_c X^P$ is the *corner locus* of PPH-polynomial P .

Example 2. *Corner locus of PPH-function.* Any $(2n-1)$ -dimensional cell Δ of triangulation \mathcal{P}_H is a common face of $2n$ -dimensional cells B^+ and B^- . Let $h \in A$ and $h_\Delta^+ = h|_{B^+}$, $h_\Delta^- = h|_{B^-}$. The ordering of the pair (B^+, B^-) sets the coorientation of Δ . The standard orientation of M and the coorientation of Δ together set the orientation of the cell Δ . Using this orientation put $(D_c X^h)_\Delta = d^c h_\Delta^+ - d^c h_\Delta^-$.

Corollary 6. *For any PPH-polynomial P*

$$(D_c X^P)_\Delta = \sum_{1 \leq i \leq q} \frac{\partial P}{\partial x_i}(h_1, \dots, h_q) (d^c(h_i)_\Delta^+ - d^c(h_i)_\Delta^-), \quad (2)$$

where 1-forms $d^c(h_i)_\Delta^\pm$ on the $(2n-1)$ -dimensional oriented cell Δ are defined in the text of example 2.

Corollary 7. *The corner locus $D_c X^P$ is a P-cycle.*

Definition 5. The P-cycle $D_c^k X^P$ is called a k -th corner locus of PPH-polynomial P .

The validity of definition 5 is based on the assertion (1) of lemma 1. Below we use the following notation:

1. $I = \{i_1, \dots, i_q : i_j \geq 0\}$, $|I| = i_1 + \dots + i_q$, $I! = i_1! \dots i_q!$, $x^I = x_1^{i_1} \dots x_q^{i_q}$.
2. If $i_p > 0$ then $I \setminus p = \{i_1, \dots, i_p - 1, \dots, i_q\}$; else $I \setminus p = \emptyset$.
3. If $I \neq \emptyset$ then $P_I(h_1, \dots, h_q) = \frac{\partial^{|I|} P}{\partial x_1^{i_1} \dots \partial x_q^{i_q}}(h_1, \dots, h_q)$; else $P_I(h_1, \dots, h_q) = 0$.
4. $\Delta \mapsto \tilde{\Delta}$ is some fixed mapping of the set of cells of triangulation \mathcal{P}_H into itself such that
 - (a) $\Delta \subset \tilde{\Delta}$
 - (b) $\dim \tilde{\Delta} = 2n$
 - (c) if $\dim \Delta = 2n$ then $\tilde{\Delta} = \Delta$.
5. H_{Δ}^i is the restriction of pluriharmonic function $(h_i)_{\tilde{\Delta}}$ to some neighborhood of the cell Δ
6. $G_{\Delta, \Gamma}^i$ is the restriction of pluriharmonic function $(h_i)_{\Gamma}$ to some neighborhood of the cell Δ , where Γ ranges over the set of $2n$ -dimensional cells containing Δ .

Lemma 1. *For any PPH-polynomial P and any $k \geq 1$*

- (1) $D_c^k X^P$ is a P-cycle.
- (2) If $\dim \Delta = 2n - k$ then

$$(D_c^k X^P)_{\Delta} = \sum_{|I|=k} P_I(h_1, \dots, h_q) Q_{I, \Delta}^k,$$

where $Q_{I, \Delta}^k$ is a polynomial of degree k in variables $\{d^c G_{\Delta, \Gamma}^i\}$ and is independent of the choice of a polynomial P .

- (3) If $|I| = k$ and $P = x^I$ then $(D_c^k X^P)_{\Delta} = I! Q_{I, \Delta}^k$.

Proof. The proof is by induction on k . For $k = 1$, all the assertions follow from corollary 6. If the assertion is true for $k - 1$ then

$$\begin{aligned}
(D_c^k X^P)_\Lambda &= \sum_{\Delta \supset \Lambda, \dim \Delta = 2n-k+1} \sum_{|I|=k-1} d^c P_I(H_\Delta^1, \dots, H_\Delta^q) Q_{I,\Delta}^{k-1} = \\
&= \sum_{\Delta \supset \Lambda, \dim \Delta = 2n-k+1} \sum_{|I|=k-1} \left(\sum_{1 \leq i \leq q} \frac{\partial P_I}{\partial x_i}(h_1, \dots, h_q) d^c H_\Delta^i \right) \wedge Q_{I,\Delta}^{k-1} = \\
&= \sum_{\Delta \supset \Lambda, \dim \Delta = 2n-k+1} \sum_{|I|=k} P_I(h_1, \dots, h_q) \sum_{1 \leq i \leq q} d^c H_\Delta^i \wedge Q_{I \setminus i, \Delta}^{k-1} = \\
&= \sum_{|I|=k} P_I(h_1, \dots, h_q) \sum_{\Delta \supset \Lambda, 1 \leq i \leq q} d^c H_\Delta^i \wedge Q_{I \setminus i, \Delta}^{k-1},
\end{aligned}$$

where the product $d^c H_\Delta^i \wedge Q_{I \setminus i, \Delta}^{k-1}$ is an odd form (as a product of even and odd forms) restricted to the cell Λ . So we can put

$$Q_{I,\Lambda}^k = \sum_{\Delta \supset \Lambda, 1 \leq i \leq q} d^c H_\Delta^i \wedge Q_{I \setminus i, \Delta}^{k-1}.$$

Assertion (2) is proved.

Applying assertion (2) to $P = x^I$ we get assertion (3).

Combining assertion (3) and proposition 1 (b) we obtain assertion (1).

Corollary 8. *If $k \geq n$ then $D_c^k X^P = 0$.*

2.3 Corner loci of PPH-cycles.

Let $i = 1, \dots, p$ and X_i be a k -dimensional P-cycle (definition 3). For PPH-polynomials P^1, \dots, P^p we define the k -cycle $X = P^1 X_1 + \dots + P^p X_p$ as $X_\Delta = P^1(X_1)_\Delta + \dots + P^p(X_p)_\Delta$.

Definition 6. The k -cycle X is called a PPH-cycle if the forms $(X_i)_\Delta$ are closed. Put $\deg X = \max_i \deg P^i$.

Corollary 9. *Any PPH-cycle is a P-cycle.*

Corollary 10. *PPH-cycles form the module over the ring of PPH-polynomials.*

Definition 7. The cycle $D_c X$ is called a corner locus of PPH-cycle X .

Proposition 2. *The corner locus $D_c X$ of any k -dimensional PPH-cycle X is a $(k - 1)$ -dimensional PPH-cycle and $\deg D_c X = \deg X - 1$.*

Proof. Let Y be a PPH-cycle of degree 0. Using the definition of operator D_c , we have

$$(D_c(h_j Y))_\Lambda = \sum_{\Delta \supset \Lambda, \dim \Delta = k} d^c H_\Delta^j \wedge Y_\Delta,$$

where the functions H_Δ^i defined in subsection 2.2. The forms $d^c H_\Delta^j \wedge Y_\Delta$ are closed. It follows that $D_c(h_j Y)$ is PHP-cycle of degree 0. Now it remains to observe that if $X = P^1 X_1 + \dots + P^p X_p$ then

$$D_c X = \sum_{1 \leq i \leq p, 1 \leq j \leq q} \frac{\partial P^i}{\partial x_j}(h_1, \dots, h_q) D_c(h_j X_i).$$

Corollary 11. *If $h \in A$ then $D_c(h^k X) = k h^{k-1} D_c(h X)$.*

2.4 PPH-cycles as currents.

Let X be a k -chain. Now suppose \bar{X} is a current such that

$$\bar{X}(\varphi) = \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_{\Delta} X_\Delta \wedge \varphi.$$

Lemma 2. *Let X be a k -cycle such that the forms X_Δ are closed. Then the current \bar{X} is closed.*

Proof.

$$\begin{aligned} d\bar{X}(\psi) &= \bar{X}(d\psi) = \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_{\Delta} X_\Delta \wedge d\psi = \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_{\Delta} d(X_\Delta \wedge \psi) = \\ &= \sum_{\Lambda \in \mathcal{P}_H, \dim \Lambda = k-1} \sum_{\Delta \supset \Lambda, \dim \Delta = k} \int_{\Lambda} X_\Delta \wedge \psi = \sum_{\Lambda} \int_{\Lambda} (\partial X)_\Lambda \wedge \psi = (\overline{\partial X})(\psi) = 0 \end{aligned}$$

Corollary 12. *Let X be a k -dimensional PPH-cycle of degree 0. Then the current \bar{X} is closed.*

Proposition 3. *Let $h \in A$, X be a k -dimensional PPH-cycle of degree 0. Then $\overline{D_c(hX)} = dd^c(h\bar{X})$.*

Proof.

$$\begin{aligned}
dd^c(h\bar{X})(\psi) &= - \sum_{\Delta \in \mathcal{P}_H, \dim \Delta = k} \int_{\Delta} hX_{\Delta} \wedge dd^c\psi = \\
&= (-1)^{k+1} \sum_{\Delta} \int_{\Delta} (d(hX_{\Delta} \wedge d^c\psi) - dh \wedge X_{\Delta} \wedge d^c\psi) = \\
&= (-1)^k \sum_{\Delta} \int_{\Delta} dh \wedge X_{\Delta} \wedge d^c\psi = (-1)^{k+1} \sum_{\Delta} \int_{\Delta} d^c h \wedge X_{\Delta} \wedge d\psi = \\
\sum_{\Delta} \int_{\Delta} d(d^c h \wedge X_{\Delta} \wedge \psi) &= \sum_{\Delta} \sum_{\Lambda \subset \Delta, \dim \Lambda = k-1} \int_{\Lambda} d^c h \wedge X_{\Delta} \wedge \psi = \overline{D_c(hX)}(\psi).
\end{aligned} \tag{3}$$

The formula (3) consists of seven equalities. We shall comment to each one.

1. Determination of the current derivative and the identity $dd^c = -d^c d$.
2. The closedness of the form X_{Δ} .
3. The Stokes formula and the k -chain hX closedness.
4. Let the form ψ bidegree is $(k-n-1, k-n-1)$ (the values of the current $dd^c(h\bar{X})$ on homogeneous components of other degrees are zero). Suppose $x \in \Delta$ and $\xi_1, \dots, \xi_{2k-2n}$ is a basis of the (real) vector space \mathbb{C}_{Δ}^x . It is easy to prove that

$$(dh \wedge d^c\psi)(\xi_1, \dots, \xi_{2k-2n}) = -(d^c h \wedge d\psi)(\xi_1, \dots, \xi_{2k-2n}).$$

$$\text{Hence } (dh \wedge X_{\Delta} \wedge d^c\psi) = -(d^c h \wedge X_{\Delta} \wedge d\psi).$$

5. The closedness of the forms $d^c h$ and X_{Δ} .
6. The Stokes formula.
7. The determination of D_c .

3 Theorem 1 (proof).

Below we use the following notation (with the notation of subsection 2.2):

1. \mathcal{B} is the ring of PPH-polynomials.
2. \mathcal{B}' is the symmetric algebra of the space A .

3. \mathcal{T} is the ring (with the unity) generated by currents $dd^c h$, where $h \in A$.
4. $\mathcal{T}' = \mathcal{T}''/I$, where \mathcal{T}'' is the symmetric algebra of the space of currents $dd^c h$, where $h \in A$, and I is the ideal generated by elements of degree $> n$.
5. $\mathcal{A}' = \mathcal{B}' \otimes \mathcal{T}'$ is the tensor product of the rings.
6. δ' is the derivation on the ring \mathcal{A}' such that

$$\delta'(h \otimes 1) = 1 \otimes dd^c h, \quad \delta'(1 \otimes dd^c h) = 0$$

for any $h \in A$.

7. $\pi: \mathcal{A}' \rightarrow \mathcal{A}$ is the ring homomorphism such that

$$h_i \otimes 1 \mapsto h_i, \quad 1 \otimes dd^c h_i \mapsto dd^c h_i$$

Theorem 1 follows from proposition 4. Proposition 4 says that the derivation δ' survives on the quotient ring \mathcal{A} of the ring \mathcal{A}' .

Proposition 4. *Let $\pi(F) = 0$; then $\pi\delta'(F) = 0$.*

Proposition 4 follows from (see below) proposition 5.

Theorem 2. (1) *For any $\tau \in \mathcal{A}$ there exists a unique PPH-cycle $X = \iota(\tau)$ such that $\tau = \bar{X}$.*

(2) *If $\tau \in \mathcal{T}$, $\deg \tau = 2k$ then $\iota(\tau)$ is a $(2n - k)$ -dimensional PPH-cycle of degree 0.*

Proof. The uniqueness of PPH-cycle is obvious.

First we prove the existence of $\iota(\tau)$ for $\tau \in \mathcal{T}$ and the assertion (2). The proof is by induction on $\deg(\tau)$.

If $\deg(\tau) = 0$ then τ is a constant function $f(x) = c$ and $\iota(\tau)$ is the $2n$ -cycle $X_\Delta = c$.

Let $\deg(\nu) = 2k - 2$ and $\tau = dd^c h \wedge \nu$, where $h \in A$. By the inductive assumption, $\iota(\nu)$ is a $(2n - k + 1)$ -dimensional PPH-cycle of degree 0.

By proposition 3, it follows that $\tau = \overline{D_c(h\nu)}$. The degree of $(2n - k)$ -dimensional PPH-cycle $D_c(h\nu)$ is 0. So we can put $\iota(\tau) = D_c(h\nu)$. The theorem for $\tau \in \mathcal{T}$ is proved.

The \mathcal{A} as \mathcal{B} -module is generated by the elements of the ring \mathcal{T} . Similarly the \mathcal{B} -module of PPH-cycles is generated by PPH-cycles of degree 0. So the map ι can be continued as the homomorphism of \mathcal{B} -modules.

Corollary 13. *If $\tau \in \mathcal{T}$ and $h \in A$ then $\iota(h\tau) = D_c(h\nu)$*

Proposition 5. $\iota\pi\delta' = D_c\iota\pi$

Proof. On elements of the ring \mathcal{T}' both parts of the required equality are zeroes. Let $\Upsilon \in \mathcal{T}'$ and $H \in \mathcal{B}'$ be an element of the first degree. If $\pi(1 \otimes \Upsilon) = \nu$ and $\pi(H \otimes 1) = h$, then $\nu \in \mathcal{T}$ and $h \in A$. Any element of the ring \mathcal{A}' is a linear combination of elements of the form $H^k \otimes \Upsilon$ (k is not fixed). So we must prove that $\iota\pi\delta'(H^k \otimes \Upsilon) = D_c\iota(h^k\nu)$.

Now using the notation from the beginning of subsection, the Leibniz product rule, and the corollaries 13 and 11, we get

$$\begin{aligned}\iota\pi\delta'(H^k \otimes \Upsilon) &= \iota\pi\delta'((H \otimes 1)^k(1 \otimes \Upsilon)) = \iota\pi(\delta'((H \otimes 1)^k)(1 \otimes \Upsilon)) = \\ \iota\pi(k(H \otimes 1)^{k-1}\delta'(H \otimes 1)(1 \otimes \Upsilon)) &= \iota\pi(k(H \otimes 1)^{k-1}(dd^c h \otimes 1)(1 \otimes \Upsilon)) = \\ \iota(kh^{k-1}\nu dd^c h) &= kh^{k-1}\iota(dd^c(h\nu)) = kh^{k-1}D_c(h\nu) = D_c\iota(h^k\nu).\end{aligned}$$

References

- [1] A. Esterov. Tropical varieties with polynomial weights and corner loci of piecewise polynomials. - Mosc. Math. J., 12:1 (2012), 55–76 (arXiv:1012.5800)
- [2] B.Kazarnovskii. On the action of complex Monge-Ampere operator on the space of piecewise linear functions. – Funktsional’nyi Analiz i ego prilozheniya (printing)
- [3] Kazarnovskii B.J. c-fans and Newton polyhedra of algebraic varieties. Izv. Math. 67 (2003), no. 3, 23-44.
- [4] B.J.Kazarnovskii. On zeroes of exponential sums. - Dokl. Akad. Nauk SSSR 257 (1981), no. 4, 804-808.
- [5] B.Ja. Kazarnovskii. Newton polyhedra and zeros of systems of exponential sums. - Funkts. Anal. Prilozh., 1984, Volume 18, Issue 4, Pages 40–49.
- [6] Alesker. S. Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. - J. Differential Geom. 63 (2003), no. 1, 63, v.95.
- [7] Semyon Alesker. Valuations on convex sets, non-commutative determinants, and pluripotential theory. - Adv. Math., 195(2):561–595, 2005.
- [8] Kazarnovskii B.Ja. Newton Polytopes, Increments, and Roots of Systems of Matrix Functions for Finite-Dimensional Representations. - Funkts. Anal. Prilozh., 2004, Volume 38, Issue 4, Pages 22–35

- [9] *Boris Kazarnovskii*. Monge-Ampere operator and tropical geometry. - Polynomial Computer Algebra, VVM Publishing, Saint Petersburg, 2011, p. 60 - 64.
- [10] *E.Bedford, B.A.Taylor*. The Dirichlet problem for a complex Monge-Ampere equations. - Invent. math., 1976, 37, N2

Institute for Information Transmission Problems, B.Karetny per. 19, 101447
Moscow, Russia
E-mail address: kazbori@iitp.ru